PROMYS India 2024 Problem Set Solutions

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Problem 8

Since it is relatively easy to run out of letters, I used a coordinate system to model the problem. I represented width of the repeating grid/tile by n. I replaced each letter by a lattice point in the first quadrant, with the bottom left A being in the position $(0,0)$. However, coordinates do not repeat in a tiled fashion like the letters are supposed to-to solve this, I considered the coordinates of a point modulo n to represent the "letter" associated with the point. The coordinates of a point relative to its parent tile is an entity analogous to the letters mentioned in the problem statement. I will continue to call them letters. Thus, the set $\{0,1,2,...,n-1\} \times \{0,1,2,...,n-1\}$ represents our complete set of letters. I named the tile at the bottom left the primary tile. The set of letters that a line though $(0,0)$ and any general point P (called a test line) on the lattice passes though is called the trace of P, denoted as $trace(P)$. I called the set of all possible traces in a lattice with tile size n the super trace of n, denoted as $superTrace(n)$. The primary objective, then, is to find how the super trace depends on n.

To begin with, I realised that not all the points in the lattice contribute uniquely to the super trace. For example, both (2, 4) and (1, 2) have the same trace; so do (3, 4) and (6, 8). Thus, I only needed to analyse points whose coordinates were co-prime to know everything about the super trace. I called such points co-prime points ((a, b), where gcd(a, b) = 1).

Theorem 1: The super trace is equal to the set of traces of all co-prime points.

My next significant realisation was that the letters which a test line passes though always appear in a repeating sequence, and the length of the sequence is always n.

Theorem 2: The trace of every point has n distinct letters.

Proof: Consider a co-prime point (a,b). Since $gcd(a,b)=1$, (a,b) will be the first point a test line from $(0,0)$ passes though. The set of all lattice points the test line will pass though is given by $\{(0,0),(a,b),(2a,2b),(3a,3b),...\}$. The set of all letters the test line passes though, then, would be $\{(0,0),(a \mod n, b \mod n),(2a \mod n, 2b \mod n),(3a \mod n, 3b \mod n),...\}$. However, this set is finite, since $(na \mod n, nb \mod n)=(0,0),((n+1)a \mod n,(n+1)b \mod n)=(a \mod n, b \mod n)$, and so on. Thus, the set of all letters becomes $S=\{(0,0),(a \mod n, b \mod n),(2a \mod n, 2b \mod n),...,((n-1)a \mod n,(n-1)b \mod n)\}$. Now, from Fermat's Little Theorem, we know that the sequence $0, a \mod n, 2a \mod n, (n-1)a \mod n$ is always a rearrangement of the set $\{0,1,2,...n-1\}$ if a and n are co-prime. With a little work, one can generalise it to show that the sequence $0, a \mod n, 2a \mod n, ...$ repeats after every $n/\gcd(a,n)$ elements, and all the $n/\gcd(a,n)$ elements are unique. Thus, the elements of S, as written, should repeat every $\operatorname{lcm}(\frac{n}{\gcd(a,n)}, \frac{n}{\gcd(b,n)})$ elements. Since $n/\gcd(a,n)$ and $n/\gcd(b,n)$ have no common factors, that expression evaluates to n. Thus, All the elements of S, as written, are unique. this implies that every co-prime point has a trace with n letters. From Theorem 1, it follows that the trace of any point must have n elements.

I proceeded to attempt to solve the question for the special case of n being prime. I realised that when this is the case, the trace of a point P also becomes the trace of every point/letter in the trace of P.

Theorem 3: $trace(Q) = trace(P) \ \ \forall\ Q \in trace(P), Q \neq (0,0)$, where n is prime.

Proof: Let $P=(a,b)$ be a co-prime point. Consider an element $Q=(ka \mod n, kb \mod n)$, $1<k<n$, from the trace of P. Now, let $ka \mod n = ka - nx_1$ and $kb \mod n = kb - nx_2$. Since $l(ka-nx_1) \equiv lka \mod n$ for any l, we get

$$trace(Q)=\{(0,0),(ka \mod n, kb \mod n),(2ka \mod n, 2kb \mod n),...,((n-1)ka \mod n,(n-1)kb \mod n)\}$$

From Fermat's Little Theorem, we know that this is merely a rearrangement of the trace of P.

From Theorem 3, it is easy to make a very useful observation.

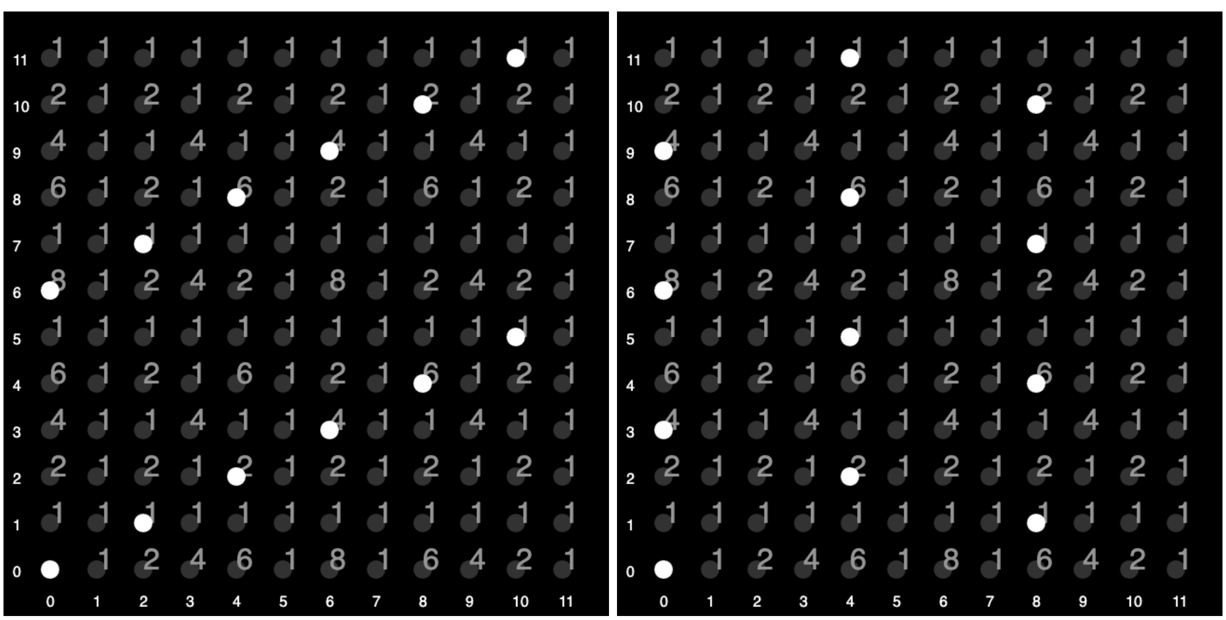
Theorem 4: No two traces in the super trace can have a common element, where n is prime (except (0, 0), which is common to all traces).

Proof: Consider two elements A and B of the super trace where n is prime. Let A and B have a common element P. Then, $trace(P) = A$, and $trace(P) = B \implies A = B$

Theorem 5: In a trace, no two points can have the same x coordinate.

Proof: Let A be a trace, with two points $P = (ka \mod n, kb \mod n)$, and $Q = (la \mod n, lb \mod n)$, $1 < k, l < n$ and Q. Then, from Fermat's Little Theorem, $ka \mod n = la \mod n \implies k = l \implies kb \mod n = lb \mod n$. This, of course, does not apply when $a = 0$.

Knowing the above, I considered the points $(1, 0), (1, 1), (1, 2), ...(1, n-1)$. All of these have the same x coordinate, thus each point must have a different trace. Since no two traces can have the same element, the traces of these n points must each grab $(n-1)$ unique letters (excluding $(0, 0)$). I also considered the point $(0, 1)$, which had all of the points of the type $(0, k)$ in its trace (None of the previous points can have these points in their trace, since they all have (0, 0) in their trace). Together, these $n+1$ points $(1, 0), (1, 1), ..., (1, n-1)$, and $(0, 1)$ each claim $(n-1)$ letters from the primary tile. In total, they claim $(n+1)(n-1) = n^2 - 1$ points for themselves. Since we initially excluded $(0, 0)$, we can add it back in to get n^2. This implies that all the points in the primary tile (in other words, all the letters), were claimed by the traces of these n+1 points. Thus, for a lattice where n is prime, the number of traces is equal to $n + 1$.
For $n = 3$, we get 4 different traces $(\{A, B, C\}, \{A, D, G\}, \{A, E, I\}, \{A, F, H\})$. The same set of letters can be obtained by passing the line through different letters, like H and F in this example. For $n = 5$ and $n = 7$, we get 6 and 8 different traces respectively.
Next, I attempted to find a general formula for the number of elements in the super trace when n is not prime. Theorem 1 and Theorem 2 still hold, but the rest do not. I discovered that the points (a, b) where $\gcd(a, n) = 1$ or $\gcd(b, n) = 1$ were always present in only one trace. Other points could be found in multiple traces. For example, when $n = 12$, the point $(4, 2)$ can be found in two different traces:



The numbers next to each point indicate how many different traces they are present it. When n is prime, they are all 1.

I have found a general formula to obtain the size of the super trace for any general n, however I have been unable to prove it yet. Let $f(n)$ denote the size of the super trace for a lattice with tile size n. I discovered that $f(p^2) = pf(p)$, where p is a prime number. For example, $f(49) = 7f(7) = 7 \times 8 = 56$. Also, $f(p_1^{r_1} p_2^{r_2} ...) = f(p_1^{r_1}) f(p_2^{r_2})...$, where $p_1, p_2, ...$ are prime.

Problem 1

Firstly, I noted that $a, b > 11$. Rearranging the equation yielded further insights:

$$\frac{1}{11} = \frac{1}{a} + \frac{1}{b} \implies 11(a+b) = ab$$

Either a or b must be divisible by 11. Since the equation is symmetric in a and b, I arbitrarily assumed a to be a multiple of 11.

$$a = 11k, k \in \mathbb{Z}^+$$
$$\implies 11k + b = kb$$
$$\implies k = \frac{b}{b-11} = 1 + \frac{11}{b-11}$$

If a has to be an integer, 11 must be divisible by $b - 11$. $b - 11 \in \{1, 11\} \implies b \in \{11, 22\}$. Thus, $(a, b) \in \{(12, 132), (22, 22), (132, 12)\}$ for $n = 11$.

For a general n, let $f = b - n$, where f is a factor of n.

$$k = 1 + \frac{n}{f}$$

$$(a, b) = \left(n + \frac{n^2}{f}, n + f\right)$$

Thus, the number of pairs (a, b), as currently defined, is equal to the number of factors of n (Notation: $fac(n)$). However, if (a, b) satisfies the equation, (b, a) will too. This makes the total number of ordered pairs (a, b) which satisfy the general equation to be $2fac(n) - 1$, where the negative one accounts for the case when $a = b$, which occurs when $f = n$.

Using the above formula, we can determine that $1/60$ can be written as a sum of two unit fractions in $fac(60) = fac(2^2 \times 3 \times 5) = 3 \times 2 \times 2 = 12$ different ways, and the number of ordered pairs (a, b) which satisfy $1/60 = 1/a + 1/b$ is 23.

Problem 3

When the hare jumps, it stretches the rubber band, and the flea's position is advanced too. If the hare jumps from a distance a from the stake and lands at a distance b from the stake, the distance of the flea from the stake will increase by a factor of b/a. Let $h(n)$ and $f(n)$ give the positions of the hare the flea respectively after both of them have made n jumps each. I worked out their values for first few jumps.

n	$h(n)$	$f(n)$
1	1000	1
2	2000	$1\left(\frac{2}{1}\right) + 1 = 3$
3	3000	$3\left(\frac{3}{2}\right) + 1 = \frac{11}{2}$
n	$1000n$	$f(n-1)\left(\frac{n}{n-1}\right) + 1$

Thus, $h(n) = 1000n$ and $f(n) = f(n-1)\left(\frac{n}{n-1}\right) + 1 \;\; \forall\, n > 1, f(1) = 1$. Writing out a couple of terms of $f(n)$ helped me to express it explicitly:

$$f(2) = \frac{2}{1} + \frac{2}{2}$$

$$f(3) = \frac{3}{2}\left(\frac{2}{1} + \frac{2}{2}\right) + 1 = \frac{3}{1} + \frac{3}{2} + \frac{3}{3}$$

$$f(4) = \frac{4}{3}\left(\frac{3}{1} + \frac{3}{2} + \frac{3}{3}\right) + 1 = \frac{4}{1} + \frac{4}{2} + \frac{4}{3} + \frac{4}{4}$$

Clearly,

$$f(n) = n\sum_{k=1}^{n} \frac{1}{k}$$

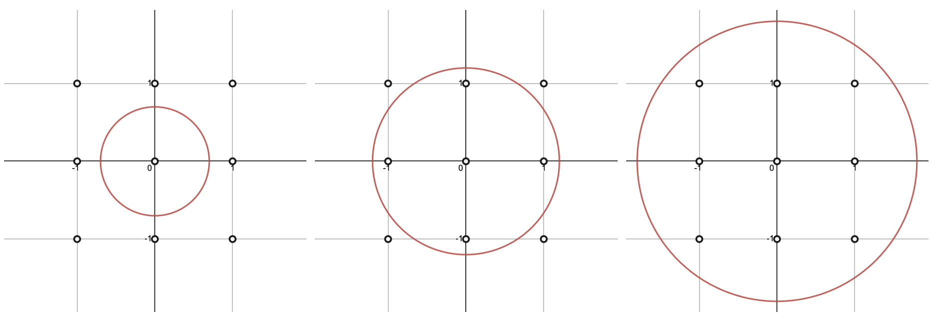
The flea will catch its meal if $f(n) \geq h(n)$, for some n.

$$\sum_{k=1}^{n} \frac{1}{k} \geq 1000$$

Since the harmonic series diverges, it will eventually become larger than 1000, given enough jumps. Thus, the flea will be able to catch the hare.

Problem 4

I started trying to solve this problem by considering a circle centered at the origin. If we increase the radius in small increments, we will notice that the lattice points will enter the interior of the circle in "batches".



The number of points which lie inside the circle form this sequence: 1, 5, 9, 13, 21, 25, 29, 37, 45, 49, 57, 61, 69, 81, 89, 97, 101, 109, 113, 121, 129, 137, 145, ... A lot of my initial work was focused on finding a pattern in these numbers, and trying to work out any adjustments to the position of the center so that the missing numbers could also be included.

However, it later occurred to me as a better idea to avoid the formation of these "batches" altogether. If we could have only one point enter the circle at a time as we slowly increased the radius, then the number of lattice points in the interior of the circle would sequentially step through all natural numbers, showing that for every positive integer n there is a circle in the plane which contains exactly n lattice points.

If only one lattice point has to enter the circle at a time when increasing the radius, the distance from the center of the circle to every lattice point must be different (points only enter in batches when they are equidistant from the center).

Let the center of the circle be located at (h, k). The distance of the center from a lattice point (x, y) is given by $\sqrt{(x-h)^2 + (y-k)^2}$. Thus, if (x_1, y_1) and (x_2, y_2) are two different lattice points, the equation

$$(x_1 - h)^2 + (y_1 - k)^2 = (x_2 - h)^2 + (y_2 - k)^2$$

must not have any solutions. Simplifying, we get

$$x_1^2 + y_1^2 - 2hx_1 - 2ky_1 = x_2^2 + y_2^2 - 2hx_2 - 2ky_2$$

The first solutions that occurred to me were irrational coordinates for the center which could not be expressed as integral multiples of each other, such as (π, e). I eventually reached a more general conclusion on the nature of such centers.

The slope of a line L connecting the origin to (h, k) will be k/h. Let a line P be perpendicular to L, and also pass through the origin. P will have a slope of $-h/k$.

Let us consider the possibilities when k/h is a rational number. Let $a/b = k/h$, where $a, b \in \mathbb{Z}^+$ and $gcd(a, b) = 1$. Since P has a slope of $-b/a$ and passes though the origin, P must also pass through the points $(a, -b)$ and $(-a, b)$. Both of these points are equidistant from the point of intersection of P and L (the origin), and hence have the same perpendicular distance from L. As a result, every point on the line L is equidistant from these two points. Therefore, if k/h is a rational, our circle can not have (h, k) as its center, since there will always exist at least two points which are equidistant from (h, k).

However, if k/h is irrational, so will be $-h/k$. With an irrational gradient, P can never pass through any two lattice points, let alone those that are equidistant from L. Hence, every lattice point is at a unique distance from (h, k).

Summary

- For every positive integer n there is a circle in a plane which contains exactly n lattice points.
- To progressively get n circles which have $1, 2, 3, \cdots, n$ lattice points inside them, one should choose a center (h, k) such that k/h is not rational, and should increase the radius from 0 in small steps such that each increment brings a new lattice point into the circle.

Problem 5

By applying rule 2 on $\frac{1}{1}$, we get $\frac{1}{2}$. Repeatedly applying rule 3 on $\frac{1}{1}$ and $\frac{1}{2}$ gives numbers of the form $\frac{n+1}{2n+1}$. These numbers are always greater than $\frac{1}{2}$, and get closer to $\frac{1}{2}$ as n increases. Let these numbers constitute a subset S_1 of S.

$$S_1 = \left\{ \frac{1}{1}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \cdots \right\}$$

The differences between the denominator and numerator of the elements of S_1 form a simple arithmetic progression: 0, 1, 2, 3, ... Thus, for any positive integer n, S_1 has an element which has n as the difference between its denominator and numerator. Now, by repeatedly applying rule 3 on an element of S_1 and $\frac{1}{1}$, we can obtain any rational number between $\frac{1}{2}$ and $\frac{1}{1}$.

For example, consider $\frac{133}{205}$. The difference between the denominator and numerator here is 72. The element in S_1 with the same difference is $\frac{73}{145}$. By repeatedly applying rule 3 on $\frac{73}{145}$ and $\frac{1}{1}$ 60 times, we get $\frac{133}{205}$. Thus, $\frac{133}{205}$ is a member of S.

Now, if any other number not between $\frac{1}{2}$ and $\frac{1}{1}$ is a part of S, it must be obtainable by the application of Rules 2 and 3 on $\frac{1}{2}$, $\frac{1}{1}$ and the elements of S_1. By definition, for any element in S_1, twice the numerator is greater than the denominator $(2n+2 > 2n+1)$. Also, the denominator is always greater than the numerator. Thus, the application of rule 2 on any element of S_1 will always yield a number between $\frac{1}{2}$ and $\frac{1}{1}$.

To conclude similarly for rule 3, let us consider two numbers $\frac{a}{b}$ and $\frac{c}{d}$, such that $\frac{1}{2} \leq \frac{a}{b}, \frac{c}{d} \leq \frac{1}{1}$. It follows that $\frac{b}{2} \leq a \leq b$ and $\frac{d}{2} \leq c \leq d$. Thus,

$$\frac{b+d}{2} \leq a + c \leq b + d$$

$$\frac{1}{2} \leq \frac{a+c}{b+d} \leq 1$$

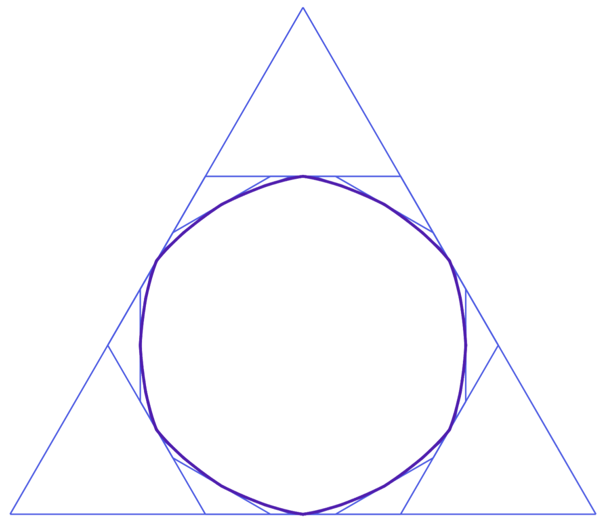
This proves that the application of rule 3 will also always yield a number between $\frac{1}{2}$ and $\frac{1}{1}$.

Therefore, S is the set of all rational numbers between and including $\frac{1}{2}$ and $\frac{1}{1}$.

$$S = \left\{ x : \frac{1}{2} \leq x \leq \frac{1}{1},\ x \in \mathbb{Q} \right\}$$

Problem 6

My first hunch was that P_∞ would be the in-circle of P_0. However, that is not possible since the polygons cease to be regular from P_2.

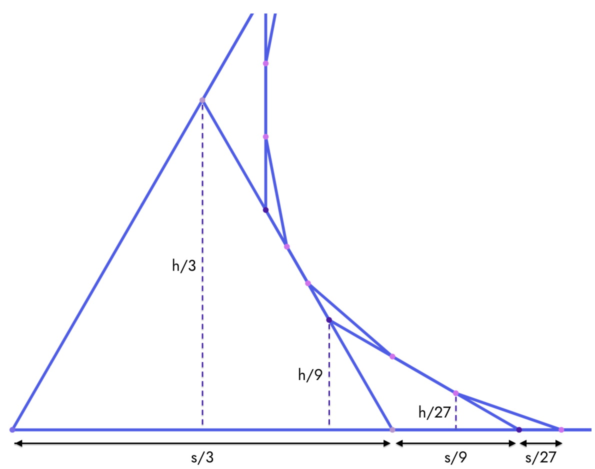


The highlighted polygon is P_9, which is a close approximation of P_∞.

Let the side length of P_0 be s, and its height be h. Let A_n denote the area of P_n. To find A_n, we should find the area that has been excluded from A_{n-1}. For example, in going from P_0 to P_1, three triangles are excluded. The three triangles are isosceles, with the equal sides measuring $s/3$ and vertex angles of $60°$. Thus,

$$A_1 = A_0 - 3\left(\frac{1}{2}\left(\frac{s}{3}\right)^2 \sin\left(\frac{\pi}{3}\right)\right)$$

However, if we wish to transform this into a general recurrence relation in A_n, we will need to account for the fact that these triangles will not always be isosceles, as a consequence of the polygons not being regular from P_2 onward. An easier approach I discovered uses the fact that both the heights and the bases of these triangles follow a simple geometric progression.



For example, when we trisect the sides of P_0, the three triangles which form the corners to be snipped off have heights $h/3$ and base lengths $s/3$ (the horizontal sides in the illustration are considered as bases). When we trisect P_1, the triangles to be snipped off now have heights $h/9$ and base lengths $s/9$ (these statements can be proved easily using the basic proportionality theorem). For P_0, we have 3 of these triangles (each corresponding to one vertex). For P_1, we have 6. Thus, to go from P_n to P_{n+1}, 3×2^n triangles must be snipped off, each of area $\frac{1}{2}\frac{s}{3^{n+1}}\frac{h}{3^{n+1}}$.

$$A_{n+1} = A_n - 3 \cdot 2^n \frac{1}{2}\left(\frac{s}{3^{n+1}}\right)\left(\frac{h}{3^{n+1}}\right)$$

$$A_{n+1} = A_n - sh\left(\frac{2^{n-1}}{3^{2n+1}}\right)$$

$$A_{n+1} = A_n - s\left(\frac{\sqrt{3}}{2}s\right)\left(\frac{2^{n-1}}{3^{2n+1}}\right)$$

$$A_{n+1} = A_n - s^2\sqrt{3}\left(\frac{2^{n-2}}{3^{2n+1}}\right)$$

Explicitly, the formula for A_n can be expressed as

$$A_n = A_0 - s^2\sqrt{3}\sum_{k=0}^{n-1}\frac{2^{k-2}}{3^{2k+1}}$$

$$A_n = A_0 - \left(\frac{4A_0}{\sqrt{3}}\right)\frac{\sqrt{3}}{12}\sum_{k=0}^{n-1}\left(\frac{2}{9}\right)^k$$

$$A_n = A_0\left(1 - \frac{1}{3}\sum_{k=0}^{n-1}\left(\frac{2}{9}\right)^k\right)$$

The terms in the sum form a converging geometric sequence with initial term 1 and common ratio $2/9$. The limiting value of such a geometric sequence is given by the formula $a/(1-r)$. Thus,

$$A_\infty = A_0\left(1 - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^k\right)$$

$$A_\infty = A_0\left(1 - \frac{1}{3}\left(\frac{9}{7}\right)\right)$$

$$A_\infty = A_0\left(\frac{4}{7}\right)$$

Summary

- The area of P_n is given by

  $$A_n = A_0\left(1 - \frac{1}{3}\sum_{k=0}^{n-1}\left(\frac{2}{9}\right)^k\right)$$

  where A_0 is the area of P_0.
- The area of P_∞ can be obtained by replacing the sum with its limiting value

  $$A_\infty = A_0\left(\frac{4}{7}\right)$$

- If $A_0 = 10$, A_1, A_2 and $Ainfty$ can be found using the above formulae.

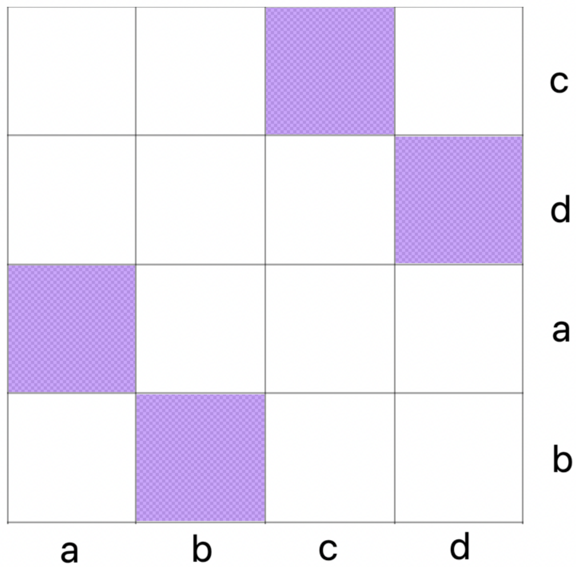
  $$A_1 = \frac{20}{3} = 6.66$$

  $$A_2 = \frac{160}{27} = 5.925925...$$

  $$A_\infty = \frac{40}{7} = 5.714285714285...$$

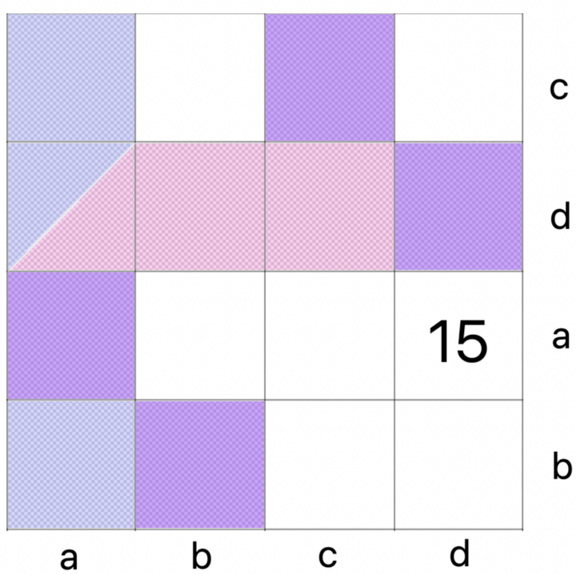
Problem 7

Let us consider a 4×4 grid to begin with. Let the numbers obtained by multiplying the numbers in each horizontal row and vertical column be a, b, c, and d. These products can be obtained in any order row-wise and column-wise. Let the squares where the column and row with the same product intersect be called "intersections". An $n \times n$ grid has n intersection squares.



The highlighted squares are intersections.

If a number is not on an intersection, it follows that other numbers with all of its factors must appear in a perpendicular row or column with respect to its vertex, so that the same product may be obtained row-wise and column-wise.



In the above example, all the squares in column one except the vertex square, and all the squares in row 3 except the vertex square must both have a factor of 15 in their product. If a number is not on an intersection, for every prime factor in its factorization, there must exist another number less than n^2 which also has said prime factor. Now, let us consider a prime number p, such that $\frac{n^2}{2} \leq p \leq n^2$. The nearest multiple of p, which is $2p$, is greater than n^2. Thus, p is the only number in the range of 1 to n^2 which has a prime factor of p. This makes it impossible for p to appear on a non intersection square.

Thus, for every prime number between $\frac{n^2}{2}$ and n^2, we must have an intersection square. But, since there are only n intersection squares available, if the number of prime numbers between $\frac{n^2}{2}$ and n^2 is greater than n, we would run out of places to place these primes, making it impossible to for the monkey to fill the grid in such a way that the cat and dog obtain the same lists of n numbers. 11 is an example of such a number. It has 13 primes(61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113) between 61 and 121.

Summary

- If the number of primes between $\frac{n^2}{2}$ and n^2 is greater than n, an arrangement where column-wise multiplication and row-wise multiplication yields the same list of numbers is not attainable.

All graphics are my own work.